

Artem G. Volosniev · Dmitri V. Fedorov ·
Aksel S. Jensen · Nikolaj T. Zinner · Manuel Valiente

Multicomponent Strongly Interacting Few-Fermion Systems in One Dimension

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Abstract The paper examines a trapped one-dimensional system of multicomponent spinless fermions that interact with a zero-range two-body potential. We show that when the repulsion between particles is very large the system can be approached analytically. To illustrate this analytical approach we consider a simple system of three distinguishable particles, which can be addressed experimentally. For this system we show that for infinite repulsion the energy spectrum is six fold degenerate. We also show that this degeneracy is partially lifted for finitely large repulsion for which we find and describe corresponding wave functions.

Keywords strongly interacting few-body systems · one dimensional harmonic traps · multicomponent fermions · Tonks-Girardeau gas

1 Introduction

Experimental study of few-body physics in cold atomic gases is a complicated task since such systems usually have a relatively large particle density and, hence, many-body correlations should be taken into account. Only very recently setups with small particle numbers were realized in Heidelberg [1; 2; 3] where ground state systems of a few fermionic atoms, ^6Li , were prepared in a quasi-one-dimensional trap. Such setups pave the way for the experimental study of few-body correlations where accurate theoretical description can be obtained through advanced numerical investigation [4; 5]. This paper overviews the newly developed method [6] to study the mentioned experimental setups in the limit of strong repulsion between particles without applying complicated numerical routines. This analytical approach gives a description of the Tonks-Girardeau gas [7; 8; 9] of a few particles in a trap. Moreover, as a by-product results obtained with this method can be used as a reference point for numerical calculations.

The structure of the paper is the following: in section 2 we introduce the Hamiltonian that is widely used to describe the relevant experimental setups [1; 3; 4; 5; 6; 10; 11], in section 3 we illustrate the approach using the simple system of three distinguishable particles in a harmonic trap, that to the best of our knowledge was not addressed before. There we find and describe eigenstates of such system in the limit of strong interparticle interaction.

Artem G. Volosniev · Dmitri V. Fedorov · Aksel S. Jensen · Nikolaj T. Zinner
Department of Physics and Astronomy, Aarhus University, DK-8000 Aarhus C, Denmark

Manuel Valiente
SUPA, Institute for Photonics and Quantum Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK
E-mail: artem@phys.au.dk

2 Formulation of the problem

This paper considers N particles of equal mass m in one spatial dimension. Additionally, the following assumptions are applied: *i*) the particles can be divided into classes of identical spinless fermions; *ii*) the system is trapped by some external potential, V_{ext} ; *iii*) the interparticle interaction is assumed to be of zero range, $V = g\delta(x_i - x_j)$, where g is a strength parameter and x_i, x_j are the coordinates of particles i and j . These assumptions lead to the following Hamiltonian

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^N V_{ext}(x_i) + g \sum_{i>j} \delta(x_i - x_j). \quad (1)$$

This model Hamiltonian is often used to study the relevant experimental setup of a few fermionic atoms ^6Li in different hyperfine states [1; 3; 4; 5; 6; 10; 11]. This model Hamiltonian allows one to explore analytical approach only in limiting cases, e.g. two particles in a harmonic trap [12] or N particles without an external confinement [13], such that the theoretical study for the relevant experiments is usually provided using different numerical techniques, e.g. [4; 5]. It is shown in [6] that one can find eigenstates for such Hamiltonian in the case of very strong repulsion between particles, i.e. if $1/g \rightarrow 0$. This regime is experimentally accessible [3] and very interesting from the theoretical point of view since it allows a theoretician to gain knowledge about strongly interacting systems.

In this paper we overview the method from [6] using a simple system of three distinguishable particles trapped in a harmonic oscillator trap. In other words we consider the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m} \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} + \frac{m\omega^2}{2} \sum_{i=1}^3 x_i^2 + g\delta(x_1 - x_2) + g\delta(x_2 - x_3) + g\delta(x_1 - x_3) \right) \Psi = E\Psi, \quad (2)$$

where Ψ is the wave function, E is the energy of the system, ω is the frequency of the harmonic oscillator, g is assumed to be large and positive, such that $1/g \rightarrow 0$. Eq. (2) can be recast into the free Schrödinger equation plus the boundary conditions

$$\left(\frac{\partial \Psi}{\partial x_i} - \frac{\partial \Psi}{\partial x_j} \right) \Big|_{x_i - x_j = +0} - \left(\frac{\partial \Psi}{\partial x_i} - \frac{\partial \Psi}{\partial x_j} \right) \Big|_{x_i - x_j = -0} = \frac{2gm}{\hbar^2} \Psi \Big|_{x_i = x_j}, \quad (3)$$

where $i, j = 1, 2, 3$ and $i \neq j$. Eqs. (2) and (3) contain all ingredients that are needed to describe a general method [6]; moreover, the system of three distinguishable particles is interesting on its own rights as experimentally relevant.

3 Method

3.1 Case of infinite repulsion between particles, $1/g = 0$.

To solve eq. (2) for large values of g we first notice that if $1/g = 0$, then eq. (3) yields $\Psi|_{x_i=x_j} = 0$, which means that eq. (2) can be seen as an equation for free particles that cannot penetrate through one another. This problem formally resembles the case of three identical spinless fermions with the only difference that now the total wave function not need to have a continuous derivative when two particles meet. This possibility to have a discontinuous derivative follows from eq. (3). However, since the particles cannot penetrate through one another it is enough to solve the Schrödinger equation only for a given ordering of the particles, i.e. $x_1 < x_2 < x_3$, with the boundary conditions $\Psi_{x_1=x_2} = \Psi_{x_2=x_3} = 0$. These boundary conditions can be satisfied only for the energies from the eigenspectrum of spinless fermions $\{E_0, E_1, \dots\}$, which implies that the total wave function for $1/g = 0$ can be built using the wave function of spinless fermions Ψ_F for a given energy E_i

$$\Psi = \begin{cases} a_1 \Psi_F & \text{for } x_2 < x_1 < x_3 \\ a_2 \Psi_F & \text{for } x_2 < x_3 < x_1 \\ a_3 \Psi_F & \text{for } x_3 < x_2 < x_1 \\ a_4 \Psi_F & \text{for } x_3 < x_1 < x_2 \\ a_5 \Psi_F & \text{for } x_1 < x_3 < x_2 \\ a_6 \Psi_F & \text{for } x_1 < x_2 < x_3 \end{cases} \quad (4)$$

where a_i are real coefficients. To obtain the wave function (4) we assumed that the energy E_i corresponds to only one wave function Ψ_F . The extension of the method for the more general situation when the energy E_i corresponds to more than one wave function is discussed in refs. [6; 14]. The possibility to build the wave function in the form of eq. (4) implies that for $1/g = 0$ the energy spectrum of eq. (2) is six fold degenerate, since all six coefficients a_i in eq. (4) are linearly independent.

3.2 Case of finitely large repulsion between particles, $1/g \rightarrow 0$.

Method. To find the behavior of the energy in the vicinity of $1/g = 0$ we use the Hellmann-Feynman theorem to obtain the derivative of the energy

$$\frac{\partial E}{\partial g} = \frac{\int dx_1 dx_2 (\Psi)_{x_3=x_1}^2 + \int dx_1 dx_3 (\Psi)_{x_2=x_3}^2 + \int dx_2 dx_3 (\Psi)_{x_1=x_2}^2}{\langle \Psi | \Psi \rangle}. \quad (5)$$

Next we notice that from eq. (3) it follows that $\Psi|_{x_i=x_j} \sim 1/g + o(1/g)$, which allows us to conclude that $E \sim E_i - K/g + o(1/g)$, where the parameter K is given by

$$K \doteq \lim_{g \rightarrow \infty} g^2 \frac{\partial E}{\partial g} = \lim_{g \rightarrow \infty} g^2 \frac{\int dx_1 dx_2 (\Psi)_{x_3=x_1}^2 + \int dx_1 dx_3 (\Psi)_{x_2=x_3}^2 + \int dx_2 dx_3 (\Psi)_{x_1=x_2}^2}{\langle \Psi | \Psi \rangle}. \quad (6)$$

The limit in this equation can be easily taken, since $(\Psi)_{x_i=x_j}^2 \sim 1/g^2 + o(1/g^2)$, which yields

$$\begin{aligned} K = & \frac{\hbar^4}{m^2} \frac{\int \left(\frac{1}{2} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} \right) |_{x_3-x_1=+0} \Psi - \frac{1}{2} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} \right) |_{x_3-x_1=-0} \Psi \right)^2 dx_1 dx_2}{\langle \Psi | \Psi \rangle} + \\ & \frac{\hbar^4}{m^2} \frac{\int \left(\frac{1}{2} \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} \right) |_{x_2-x_3=+0} \Psi - \frac{1}{2} \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} \right) |_{x_2-x_3=-0} \Psi \right)^2 dx_1 dx_3}{\langle \Psi | \Psi \rangle} + \\ & \frac{\hbar^4}{m^2} \frac{\int \left(\frac{1}{2} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) |_{x_1-x_2=+0} \Psi - \frac{1}{2} \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) |_{x_1-x_2=-0} \Psi \right)^2 dx_2 dx_3}{\langle \Psi | \Psi \rangle}. \end{aligned} \quad (7)$$

Using the wave function (4) the value of K can be written as

$$K = \gamma \frac{(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + (a_4 - a_5)^2 + (a_5 - a_6)^2 + (a_1 - a_6)^2}{a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2}, \quad (8)$$

where γ is defined as

$$\gamma = \frac{\hbar^4}{m^2} \frac{\int_{x_2 < x_3} \left(\frac{\partial}{\partial x_1} \Psi_F \right)^2 |_{x_1=x_3} dx_2 dx_3}{\int_{x_1 < x_2 < x_3} \Psi_F^2 dx_1 dx_2 dx_3}. \quad (9)$$

To determine the behavior of the energy we need to find the coefficient K . To do so we use the variational treatment, i.e. we minimize the energy by varying K with respect to the coefficients a_i .

$$\frac{\partial K}{\partial a_i} = 0. \quad (10)$$

In this way we find the wave functions (defined by a_i and eq. (4)) to which the states outside of the degenerate point ($1/g = 0$) are adiabatically connected. This procedure is variational, since it relies on varying K with respect to the coefficients a_i . However, it produces an exact solution, since the wave function Ψ for $1/g = 0$ can always be written in the form (4), so we vary in the full space.

Solution. Equation (10) produces the system of linear equations

$$\begin{aligned} -a_6 + 2a_1 - a_2 &= K a_1 / \gamma, \\ -a_{i-1} + 2a_i - a_{i+1} &= K a_i / \gamma, \text{ for } i = 2, 3, 4, 5 \\ -a_5 + 2a_6 - a_1 &= K a_6 / \gamma. \end{aligned} \quad (11)$$

This can be seen as a usual eigenvalue problem for a system of linear equations, which can be easily solved using standard techniques from linear algebra. The solution contains six eigenstates:

$$\begin{aligned}
K_1 = 4\gamma &\rightarrow a_2 = -a_1, a_3 = a_1, a_4 = -a_1, a_5 = a_1, a_6 = -a_1 \\
K_2 = 3\gamma &\rightarrow a_2 = -a_1/2, a_3 = -a_1/2, a_4 = a_1, a_5 = -a_1/2, a_6 = -a_1/2 \\
K_3 = 3\gamma &\rightarrow a_2 = -2a_1, a_3 = a_1, a_4 = a_1, a_5 = -2a_1, a_6 = a_1 \\
K_4 = \gamma &\rightarrow a_2 = a_1, a_3 = 0, a_4 = -a_1, a_5 = -a_1, a_6 = 0 \\
K_5 = \gamma &\rightarrow a_2 = 0, a_3 = -a_1, a_4 = -a_1, a_5 = 0, a_6 = a_1 \\
K_6 = 0 &\rightarrow a_2 = a_1, a_3 = a_1, a_4 = a_1, a_5 = a_1, a_6 = a_1
\end{aligned}$$

where the coefficient a_1 is left for the overall normalization. First of all one notices that the lowest energy state defined by K_1 has a fully symmetric wave function and can be obtained from the Fermi-Bose mapping, which is first discussed in ref. [8]. This is always the case for the Hamiltonians in the form of eq. (1), without any symmetry requirements. The state with $K_6 = 0$ corresponds to a fully antisymmetric wave function of the system of spinless fermions. Four other states are the same as for the system of two spinless fermions and a third particle [4; 5; 6]. The degeneracy of the spectrum, $K_2 = K_3$ and $K_4 = K_5$, comes from the observation that one can antisymmetrize either x_1 and x_2 particles or x_1 and x_3 particles. Combinations of states with K_2, K_3, K_4, K_5 also appear for two spinless bosons and a third particle as shown in [15].

In this way we obtained the total energy spectrum of the system of three strongly interacting particles, where six states can be divided into three classes: three spinless bosons, three spinless fermions, and two spinless fermions interacting with a third particle.

4 Conclusions

In this paper one-dimensional strongly interacting systems of particles were investigated. We presented the method to treat such systems with zero-range repulsive interactions of large strength. To illustrate the approach we analyzed a simple system of three particles. For such a system we found the energy spectrum and the corresponding wave functions.

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